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A NEW CLASS OF $\mathscr{L}^1$-SPACES

BY

J. BOURGAIN

ABSTRACT

It is shown that the class of separable $\mathscr{L}^1$-spaces with the Radon-Nikodym property only admits L^1 as universal element.

1. Introduction

A $\mathscr{L}^1$-space is a Banach space whose local structure is like l^1. More precisely, we say that X is a $\mathscr{L}^1_\lambda$-space ($\lambda \geqq 1$) provided any finite dimensional subspace E of X is contained in a finite dimensional subspace F of X which satisfies $d(F, l^1(n)) \leqq \lambda$, where $n = \dim F$ and d means the Banach-Mazur distance.

It is well-known that a complemented subspace of an $L^1(\mu)$-space is a $\mathscr{L}^1$-space. An important open question is whether or not these spaces are also isomorphic to L^1-spaces. The answer is affirmative in case of a norm-1 projection (see [18]) or if the given subspace has the Radon-Nikodym property (see [8]). Conversely, it is untrue that $\mathscr{L}^1$-spaces are always isomorphic to complemented subspaces of an L^1-space. Recently, joint work of W. B. Johnson and J. Lindenstrauss (see [12]) led to a class of $\mathscr{L}^1$-spaces which are Schur, Radon-Nikodym and don't embed in a separable dual space. At the same time, they solve positively the problem on the existence of a continuum number of mutually non-isomorphic separable $\mathscr{L}^1$-spaces. If $\mathscr{H}$ is a class of Banach spaces, we say that the space B is universal for $\mathscr{H}$ provided any member of $\mathscr{H}$ is isomorphic to a subspace of B. In this paper, the existence is shown of a family $\mathscr{H}$ of separable $\mathscr{L}^1$-spaces satisfying the Radon-Nikodym property, such that any separable Banach space universal for $\mathscr{H}$ contains a copy of $L^1[0,1]$. This leads to an extension of some earlier work on $\mathscr{L}^p$-spaces for $1 < p < \infty$ and solves a question raised by A. Pelczynski (cf. [6]).

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2. A construction technique for $\mathscr{L}^1$-spaces

In this section, we use the notation L^1 to denote a separable $L^1(\mu)$-space. Our starting point is a simple and rather general way of constructing $\mathscr{L}^1$-spaces starting from certain operators on L^1. More precisely, we will prove the following

THEOREM 1. *Assume $T: L^1 \to L^1$ an operator and E a subspace of L^1 on which T is the identity, i.e. $T(f) = f$ for each $f \in E$. Then*

(a) *If T does not fix an L^1-copy, then E can be embedded in a $\mathscr{L}^1$-space not containing L^1. We recall that an operator is L^1-fixing provided it is an isomorphism when restricted to a subspace isomorphic to L^1.*

(b) *Let T be such that whenever $S: L^1(\mu) \to L^1$ is an operator, the representability of $(I - T)S$ implies the representability of S. Then E can be embedded in a $\mathscr{L}^1$-space with the Radon–Nikodym property.*

PROOF. Fixing $\rho > 1$, one can find a sequence of subspaces U_i of L^1 satisfying the following conditions:

(1) Each U_i is finite dimensional, let us say $d_i = \dim U_i$,

(2) $d(U_i, l^1(\dim U_i)) < \rho$,

(3) $U_i \subset U_{i+1}$,

(4) $T(U_i) \subset U_{i+1}$,

(5) $\bigcup_{i=1}^{\infty} U_i$ is dense in L^1,

(6) $\bigcup_{i=1}^{\infty} (E \cap U_i)$ is dense in E.

That this can be done is straightforward and we let the reader check the details.

In what follows, $\oplus$ will denote the direct sum in l^1-sense. Define

$$\mathscr{X} = L^1 \oplus \bigoplus_{i=1}^{\infty} U_i$$

and let $P: \mathscr{X} \to L^1$ and $P_i: \mathscr{X} \to U_i$ be the natural projections. We further introduce for each j the space

$$\mathscr{X}_j = U_j \oplus \bigoplus_{i=1}^{j} U_i$$

which embeds in $\mathscr{X}$ in a natural way.

For fixed j, let $I_j: \mathscr{X}_j \to \mathscr{X}$ be the operator defined as follows:

$$\begin{cases} PI_j(x) = TP(x), \\ P_iI_j(x) = P_i(x) \qquad \text{for } i = 1, \cdots, j, \\ P_{j+1}I_j(x) = P(x) - TP(x) - \sum_{i \leq j} P_i(x), \\ P_iI_j(x) = 0 \qquad \text{for } i > j+1, \end{cases}$$

which makes sense by conditions (3) and (4) on the spaces U_i. Notice that

$$(*) \qquad P(x) = (P + \Sigma_i P_i) I_j(x)$$

for all $x \in \mathscr{X}_j$

Since the inequalities

$$\tfrac{1}{2}\|x\| \leqq \|I_j(x)\| \leqq 2(1 + \|T\|)\|x\|$$

are clearly satisfied, we see that I_j is an isomorphism on its range B_j.

More precisely, we have

$$d(B_j, \mathscr{X}_j) \leqq 4(1 + \|T\|)$$

and thus

$$d(B_j, l^1(d_j)) \leqq 4\rho(1 + \|T\|).$$

Our next claim is that B_j is a subspace of B_{j+1}. Let indeed $x \in \mathscr{X}_j$ and define y by

$$\begin{cases} P(y) = P(x), \\ P_i(y) = P_i(x) \qquad \text{for } i = 1, \cdots, j, \\ P_{j+1}(y) = P(x) - TP(x) - \sum_{i \leq j} P_i(x), \\ P_i(y) = 0 \qquad \text{for } i > j+1, \end{cases}$$

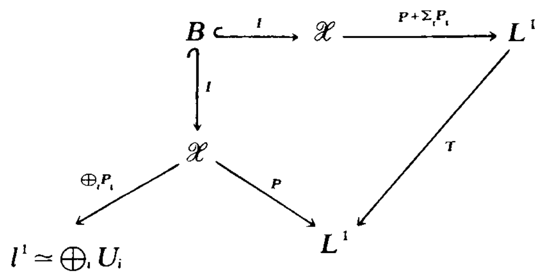
which is a member of $\mathscr{X}_{j+1}$. A simple verification shows that $I_{j+1}(y) = I_j(x)$. This shows that $B_j \subset B_{j+1}$.

Let $B = \overline{\bigcup_{j=1}^{\infty} B_j}$. L^1 embeds in $\mathscr{X}$ by identification with the first coordinate. By hypothesis on T, one has that $I_j(x) = x$ whenever $x \in E \cap U_j \hookrightarrow \mathscr{X}_j$. Thus $E \cap U_j$ is a subspace of B_j and we conclude that E is a subspace of B.

We show that L^1 does not embed in B if T does not fix an L^1-copy. As a consequence of (*), we get

(**) $$P(x) = T\Big(P + \sum_i P_i\Big)(x) \qquad \text{for all } x \in B$$

leading to the following scheme:

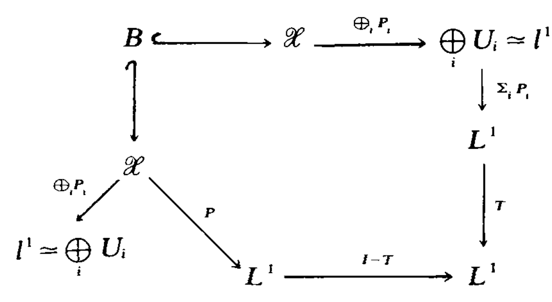


If B contains an L^1-subspace, one of the operators $\bigoplus_i P_i \circ I$, $P \circ I$ has to fix a copy of L^1 (see [9]). Since $\bigoplus_i P_i \circ I$ ranges in $\bigoplus_i U_i$ which is isomorphic to l^1 and $P \circ I$ factors over T, T will fix an L^1-space. This proves (a) of the Theorem.

In order to show (b) we rewrite (**) in the form

(***) $$(I - T)P(x) = T\Big(\sum_i P_i\Big)(x) \qquad \text{for all } x \in B$$

which gives the diagram



Suppose now $S : L^1(\mu) \to B$ is a non-representable operator. Since $S = PS \oplus (\bigoplus_i P_i)S$ and $\bigoplus_i P_i$ ranges in l^1, it follows that PS will be non-representable. But, under hypothesis (b) of the theorem, this implies the non-representability of the operator $(I - T)PS$ factoring through l^1, a contradiction.

So the proof of Theorem 1 is complete.

Theorem 1 will be applied to certain spaces E and operators T which will be described in the next two sections.

3. Treespaces

Our aim is to recall briefly the definition and basic properties of certain subspaces of L^p $(1 \leqq p \leqq \infty)$ which were introduced and discussed in [6].

$\mathbf{N}$ will be the set of natural numbers $1, 2, 3, \cdots$. Denote by $\mathscr{R}$ the set of finite complexes of integers, thus $\mathscr{R} = \bigcup_{k=1}^{\infty} \mathbf{N}^k$. For $c \in \mathscr{R}$, $|c|$ is the length of c. If $c \in \mathscr{R}$, $d \in \mathscr{R}$, we write $c < d$ provided d starts with c. This leads to a partial order on $\mathscr{R}$. Let us call a branch-set any subset of $\mathscr{R}$ whose members are mutually comparable.

A tree T (on $\mathbf{N}$) is a subset of $\mathscr{R}$ with the property that a predecessor of a member of T is still a member of T. We say that a tree T is well-founded provided there is no sequence $n_1, n_2, \cdots, n_k, \cdots$ such that $(n_1, n_2, \cdots, n_k) \in T$ for each k. If T is well-founded, denote $\mathrm{o}[T]$ the (countable) ordinal of T. For the definition of $\mathrm{o}[T]$, the reader may consult [7] or also [6].

We define G as the product group $\{1, -1\}^{\mathscr{R}}$ equipped with its Haar-measure m, which is simply the product measure $\bigoplus_{c \in \mathscr{R}} m_c$, where $m_c(1) = \frac{1}{2} = m_c(-1)$ for each $c \in \mathscr{R}$. Obviously G can be identified with the Cantor group. The characters of G are the Walsh functions $w_s = \Pi_{c \in S} r_c$ where S is any finite subset of $\mathscr{R}$ and the c-Rademacker function r_c is defined by $r_c(x) = x_c$.

Our next goal is to define the translation-invariant subspaces X_T^p of $L^p(G)$ for $1 \leqq p \leqq \infty$.

Definition. For $1 \leqq p \leqq \infty$ and T a tree on $\mathbf{N}$, take X_T^p the closed linear span in $L^p(G)$ of the Walsh-functions w_S where S is a branch-set and $S \subset T$.

We will now summarize the basic properties of these spaces X_T^p, which are established in [6].

Theorem 2. (1) *Assume T well-founded, then*

(a) *The spaces X_T^1 have the Radon–Nikodym property.*

(b) *For $1 < p < \infty$, the spaces X_T^p don't contain $L^p[0,1]$ isomorphically.*

(c) *The spaces X_T^∞ have both the Radon–Nikodym property and the Schur property.*

(2) *If $1 \leqq p < \infty$ (resp. $p = \infty$) and B is a separable Banach space which is universal for the family $\{X_T^p$; T well-founded tree$\}$, then $L^p[0,1]$ (resp. $C[0,1]$) embeds in B.*

(3) *For fixed $1 < p < \infty$, the spaces X_T^p are uniformly complemented in $L^p(G)$ by the orthogonal projection and consequently either Hilbert-spaces or $\mathscr{L}^p$-spaces* (cf. [16]).

(3) does not hold anymore for $p = 1$. More precisely, if T is non-finite, then the spaces X_T^1 are uncomplemented in $L^1(G)$ and they are never $\mathscr{L}^1$-spaces. Our aim is to show however that

THEOREM 3. *For T well-founded, the space X_T^1 is a subspace of a $\mathscr{L}^1$-space with the Radon–Nikodym property.*

In view of Theorem 2, this leads to the following result:

COROLLARY 4. *Any separable Banach space which is universal for the class of separable Radon–Nikodym $\mathscr{L}^1$-spaces contains a copy of L^1. In particular, the latter class of spaces has no universal element.*

The construction of these envelopping $\mathscr{L}^1$-spaces will be done by using the result presented in the second section of this paper. More precisely, we will prove the following

PROPOSITION 5. *For any well-founded tree T, there is an operator on $L^1(G)$ which is the identity on X_T^1 and satisfies condition* (b) *of Theorem* 2.

From this fact and Theorem 2, Theorem 3 follows.

4. Construction of certain measures on G

Denote by $C(G)$ the space of continuous functions on G. If μ is a measure on G and S a finite subset of $\mathscr{R}$, the S-Fourier coefficient $\hat{\mu}(S)$ of μ is given by $\hat{\mu}(S) = \int_G w_S d\mu$.

If μ and ν are two measures on G, the convolution $\mu * \nu$ is defined by $(\mu * \nu)(f) = \int f(x \cdot y)\mu(dx)\nu(dy)$ for $f \in C(G)$.

For a subset S of $\mathscr{R}$, $\mathscr{E}_S$ will denote the conditional expectation with respect to the sub-σ-algebra generated by the Rademacker functions $(r_c)_{c \in S}$.

If μ is a measure on G and S a subset of $\mathscr{R}$, we say that μ is S-dependent provided $\mu = \mu \circ \mathscr{E}_S$.

For S a subset of $\mathscr{R}$, take $\tilde{S} = \{c \in \mathscr{R}; c < d \text{ for some } d \in S\}$. For convenience, we introduce the "empty complex" (cf. [7]), denoted by the symbol $\emptyset$. Define $\mathscr{R}^* = \mathscr{R} \cup \{\emptyset\}$.

If $c \in \mathscr{R}$, let $c' \in \mathscr{R}^*$ be the immediate predecessor of c.

A weightfunction will be a function τ on $\mathscr{R}^*$ ranging in the open interval $]0,1[$ such that $K_\tau = \prod_{c \in \mathscr{R}^*} (2\tau(c)^{-1} - 1) < \infty$.

If τ is a weightfunction, take $[\tau; \emptyset] = 1$ and $[\tau; S] = \prod_{c \in \tilde{S}} \tau(c')$, if S is a nonempty finite subset of $\mathscr{R}$.

The next lemma is needed for a later purpose.

LEMMA 6. *Let τ be a weightfunction, $\varepsilon > 0$ and define*

$$\mathscr{S}_\varepsilon = \{S \subset \mathscr{R};\ S \text{ finite and } [\tau; S] > \varepsilon\}.$$

Assume further (S_k) a sequence of disjoint finite subsets of $\mathscr{R}$ so that for each k one can find $S \in \mathscr{S}_\varepsilon$ with $S \cap S_l \neq \emptyset$ for each $l = 1, \cdots, k$. Then there exists a sequence (n_i) of integers such that $(n_1, \cdots, n_j) \in \bigcup_k \tilde{S}_k$ for each j.

PROOF. Let us first remark that for fixed $c \in \mathscr{R}^*$ we find $[\tau; S] \leqq \tau(c)^{|M|}$ for each $S \subset \mathscr{R}$, where $M = \{n \in \mathbf{N};\ (c, n) < d$ for some $d \in S\}$.

Since $\tau(c) < 1$, this implies that $|M|$ is uniformly bounded for S ranging in $\mathscr{S}_\varepsilon$.

Since the S_k are finite sets, one can pick elements $c_k \in S_k$ such that for all k there is some $S \in \mathscr{S}_\varepsilon$ with $\{c_1, c_2, \cdots, c_k\} \subset S$. It is now possible to construct a sequence of integers (n_i), so that for each j the set $\{k; (n_1, \cdots, n_j) < c_k\}$ is infinite. We indicate briefly the inductive procedure. Suppose $n_1, \cdots, n_j$ were obtained. By hypothesis, any finite subset of the set $\{(n_1, \cdots, n_j, n);\ n \in N\}$, where $N = \{n \in \mathbf{N};\ (n_1, \cdots, n_j, n) < c_k$ for some $k\}$, is contained in $\tilde{S}$ for some $S \in \mathscr{S}_\varepsilon$. By the first observation we made, we conclude that N is finite. This allows us to choose n_{j+1} so that $\{k; (n_1, \cdots, n_j, n_{j+1}) < c_k\}$ is again infinite.

Clearly, for each j, $(n_1, n_2, \cdots, n_j) \in \tilde{S}_k$ for some k and this ends the proof.

The main objective of this section is to prove the following result.

PROPOSITION 7. *Let τ be a weightfunction. Then there exists a measure μ on G satisfying the following properties:*

(1) $\|\mu\| \leqq K_\tau$.

(2) *$\hat{\mu}(S) = 1$ if S is a branch set.*

(3) *If S is a finite subset of $\mathscr{R}$ and $f \in C(G)$ is $(\mathscr{R}\backslash S)$-dependent then* $|\int w_S(x) f(x) \mu(dx)| \leqq K_\tau [\tau; S] \|f\|_\infty$.

PROOF. For $c \in \mathscr{R}^*$, let

$$\mathscr{R}_c^* = \{d \in \mathscr{R}^*; c < d\} \quad \text{and} \quad \mathscr{R}_c = \{d \in \mathscr{R}; c < d\}.$$

We let δ be the Dirac-measure on G and δ_c the Dirac-measure on the c-factor $\{1, -1\}$ in the product G.

If for fixed $c \in \mathscr{R}$, we define the measure ν_c on G by

$$\nu_c = \tau(c')\delta + (1 - \tau(c'))\Big(\bigotimes_{d \in \mathscr{R}_c^*} m_d \otimes \bigotimes_{d \in \mathscr{R}\backslash\mathscr{R}_c^*} \delta_d\Big)$$

then clearly

(4) $\|\nu_c\| = 1$.

(5) $\hat{\nu}_c(S)=1$ if $c \notin \tilde{S}$ and $\hat{\nu}_c(S)=\tau(c')$ if $c \in \tilde{S}$.

(6) If S is a finite subset of $\mathcal{R}$ and $f \in C(G)$ is $(\mathcal{R}\setminus S)$-dependent, then $|\int w_S(x)f(x)\nu_c(dx)| \leqq \hat{\nu}_c(S)\|f\|_\infty$.

Let ν be the measure on G obtained by convolution of the ν_c, thus

$$\nu = \underset{c\in\mathcal{R}}{\ast} \nu_c.$$

As an easy verification shows, the following holds:

(7) $\|\nu\|=1$.

(8) $\hat{\nu}(S)=[\tau;S]$.

(9) If S is a finite subset of $\mathcal{R}$ and $f \in C(G)$ is $(\mathcal{R}\setminus S)$-dependent, then $|\int w_S(x)f(x)\nu(dx)| \leqq [\tau;S]\|f\|_\infty$.

Next, consider for fixed $c \in \mathcal{R}^*$ the following measure η_c on G:

$$\eta_c = \tau(c)^{-1}\delta + (1-\tau(c)^{-1})\Big(\bigotimes_{d\in\mathcal{R}_c} m_d \otimes \bigotimes_{d\in\mathcal{R}\setminus\mathcal{R}_c} \delta_d\Big)$$

satisfying

(10) $\|\eta_c\| \leqq 2\tau(c)^{-1}-1$.

(11) $\hat{\eta}_c(S)=1$ if $\mathcal{R}_c \cap S=\emptyset$ and $\hat{\eta}_c(S)=\tau(c)^{-1}$ if $\mathcal{R}_c\cap S\neq\emptyset$.

Since τ is a weightfunction, we can define the convolution η of the η_c, thus

$$\eta = \underset{c\in\mathcal{R}^*}{\ast} \eta_c$$

for which

(12) $\|\eta\| \leqq K_\tau$.

Finally, take $\mu=\nu*\eta$. Then clearly $\|\mu\|\leqq\|\nu\|\,\|\eta\|\leqq K_\tau$.

In order to verify (2), let S be a finite branch set. Then $\hat{\mu}(S)=\hat{\nu}(S).\hat{\eta}(S)=[\tau;S].\Pi\{\tau(c)^{-1}; c\in\mathcal{R}^* \text{ with } \mathcal{R}_c\cap S\neq\emptyset\}=[\tau;S].\Pi_{c\in S}\tau(c')^{-1}=1$, as required.

Let us check (3). So take S a finite subset of $\mathcal{R}$ and an $(\mathcal{R}\setminus S)$-dependent function $f\in C(G)$. We have

$$\int w_S(x)f(x)\mu(dx)=\int w_S(x.y)f(x.y)\nu(dx)\eta(dy)$$

and thus

$$\begin{aligned}\Big|\int w_S(x)f(x)\mu(dx)\Big| &\leqq \int\Big|\int w_S(x)f(x.y)\nu(dx)\Big|\,|\eta|(dy)\\ &\leqq \|\eta\|[\tau;S]\sup_y\|f_y\|_\infty\\ &\leqq K_\tau[\tau;S]\|f\|_\infty\end{aligned}$$

completing the proof.

5. Non-representable operators ranging in L^1-product spaces

Our purpose is to present here some technical ingredients needed for the next section.

The following result is essentially known, but we include its proof here for selfcontainedness.

LEMMA 8. *Let μ, ν be probability spaces and $S: L^1(\mu) \to L^1(\nu)$ a non-representable operator. Then there exist a bounded convex subset C of $L^1(\nu)$ and $\rho > 0$ such that the following holds whenever $T: L^1(\nu) \to B$ is an operator with TS representable: If $f \in C$ and $\delta > 0$, then there exists some $g \in C$ so that $\|T(f-g)\| < \delta$ and $\int_A |g| d\nu \geqq \rho$ for some ν-measurable set A with $\nu(A) < \delta$.*

PROOF. If S is not representable, then one can find a μ-measurable set Ω with $\mu(\Omega) > 0$ and $\rho > 0$ such that for any $\Omega' \subset \Omega$, $\mu(\Omega') > 0$ and $\delta > 0$, there exists $\Omega'' \subset \Omega'$, $\mu(\Omega'') > 0$ with $\int_A |S(\Omega'')| d\nu > \rho\mu(\Omega'')$ for some A with $\nu(A) < \delta$ (cf. [8]). If now $\Omega_1, \cdots, \Omega_d$ are subsets of Ω with positive measure and $\delta > 0$, there exist subsets $\Omega_i' \subset \Omega_i$ $(1 \leqq i \leqq d)$ of positive measure satisfying the following condition:

There exists a set A with $\nu(A) < \delta$ such that

$$\int_A \left| \sum_i a_i S(\Omega_i') \right| d\nu \geqq \rho \sum_i |a_i| \mu(\Omega_i')$$

for all scalars $a_1, \cdots, a_d$.

The proof of the latter fact is elementary and left as an exercise to the reader.

We show that

$$C = \{S(\varphi); \varphi \in L^1_+(\Omega), \int \varphi du = 1\}$$

satisfies the condition of the lemma.

So fix $f = S(\varphi)$, $\varphi \in L^1_+(\Omega)$, $\int \varphi d\mu = 1$ and $\delta > 0$.

Using the representability of the operator TS, it is possible to find a partition $\Omega_1, \cdots, \Omega_d$ of Ω and scalars $a_1, \cdots, a_d$, such that

(i) $\|TS(\Omega_i)/\mu(\Omega_i) - TS(\Omega')/\mu(\Omega')\| < \tau$, whenever $\Omega' \subset \Omega_i$, $\mu(\Omega') > 0$,

(ii) $\|\varphi - \sum_i a_i \chi_{\Omega_i}\|_1 < \tau$,

(iii) $\sum_i a_i \mu(\Omega_i) = 1$,

where $\tau = \delta/(1 + \|TS\|)$.

By the previous observation, one can obtain subsets $\Omega_i' \subset \Omega_i$ $(1 \leqq i \leqq d)$ of positive measure and a set A with $\nu(A) < \delta$, so that

$$\int_A \left| \sum_i a_i \frac{\mu(\Omega_i)}{\mu(\Omega_i')} S(\Omega_i') \right| d\nu \geqq \rho \sum_i a_i \mu(\Omega_i) = \rho.$$

Take

$$\psi = \sum_i a_i \frac{\mu(\Omega_i)}{\mu(\Omega_i')} \chi_{\Omega_i'}$$

and $g = S(\psi)$, belonging to C. Thus $\int_A |g| d\nu \geqq \rho$ and

$$\begin{aligned} \| T(f-g) \| &= \| TS(\varphi) - TS(\psi) \| \\ &\leqq \tau \| TS \| + \sum_i a_i \left\| TS(\Omega_i) - \frac{\mu(\Omega_i)}{\mu(\Omega_i')} TS(\Omega_i') \right\| \\ &< (1 + \| TS \|)\tau, \end{aligned}$$

completing the proof.

COROLLARY 9. *Under the hypothesis of Lemma 8, one can find $C \subset L^1(\nu)$ and $\rho > 0$ such that whenever $T : L^1(\nu) \to L^1(\nu)$ is an operator with TS representable, $f \in C$ and $\delta > 0$, there exists $g \in C$ satisfying $\|f - g\| > \rho$, $|\int (f - g) d\nu| < \delta$ and $\|T(f-g)\| < \delta$.*

Assume now $(\Omega_i, \nu_i)_{i \in D}$ a family of probability spaces and consider the product space $(\Omega, \nu) = (\Pi_i \Omega_i, \otimes_i \nu_i)$. For $E \subset D$, denote $\mathscr{E}_E$ the corresponding conditional expectation. We claim the following

LEMMA 10. *If $S : L^1(\mu) \to L^1(\nu)$ is non-representable, then D has a finite subset E so that $(I - \mathscr{E}_{D \setminus E}) . S$ is non-representable too.*

PROOF. Let $C \subset L^1(\nu)$ and $\rho > 0$ be as in the preceding corollary. Assume the above statement wrong. Successive applications of (8) allow us then to construct a sequence (f_k) in C and an increasing sequence (E_k) of finite subsets of D in such a way that

(1) $\|f_k - \mathscr{E}_{E_k}[f_k]\|_1 < 2^{-k}$,
(2) $\|f_k - f_{k+1}\|_1 > \rho$,
(3) $|\int (f_k - f_{k+1}) d\nu| < 2^{-k}$,
(4) $\|(f_k - f_{k+1}) - \mathscr{E}_{D \setminus E_k}[f_k - f_{k+1}]\|_1 < 2^{-k}$.

Take then $\varphi_1 = \mathscr{E}_{E_1}[f_1]$ and $\varphi_k = \mathscr{E}_{E_k \setminus E_{k-1}}[f_k - f_{k-1}]$ for $k > 1$. By (1) and (4), we find that $\|f_k - f_{k-1} - \varphi_k\|_1 < 8 . 2^{-k}$. Consequently, by (2), $\|\varphi_k\|_1 > \rho - 8.2^{-k}$ and, by (3), $|\int \varphi_k d\nu| < 10 . 2^{-k}$.

Consider now the sequence $\psi_k = \varphi_k - \int \varphi_k d\nu$, which consists of independent mean-zero functions. Since (ψ_k) is an unconditional basic sequence in $L^1(\nu)$, (ψ_k) is also boundedly complete (cf. [18]). But $\|\psi_k\| \geqq \rho - 18.2^{-k}$ and on the other hand

$$\left\| \sum_{k=1}^{n} \psi_k \right\|_1 \leqq 2 \left\| \sum_{k=1}^{n} \varphi_k \right\|_1 \leqq 2\|f_n\|_1 + 8 \sum_{k=1}^{n} 2^{-k} < \sup\{\|f\|_1; f \in C\} + 8,$$

a contradiction.

Repeating applications of Lemma 10 leads to the next

COROLLARY 11. *Suppose moreover* (Ω_i, ν_i) *purely atomic for each* $i \in D$. *Then, under the hypothesis of Lemma 9, there is a sequence* (E_k) *of disjoint finite subsets of* D, *so that for all* k *the operator* $\prod_{l=1}^{k} (I - \mathscr{E}_{D \setminus E_l}) \circ S$ *is not representable.*

6. Application to certain operators on $L^1(G)$

Referring to section 4, let τ be a fixed weightfunction and let μ be the measure on G constructed in Proposition 7. Consider the operator Λ on $L^1(G)$ obtained by μ-convolution, i.e.

$$\Lambda(f)(x) = \int_G f(x.y)\mu(dy).$$

The following is easily derived from Proposition 7.

PROPOSITION 12. (1) $\|\Lambda\| \leqq K_\tau$.

(2) $\Lambda(w_S) = w_S$ *if* S *is a branch.*

(3) *If* S *is a finite subset of* $\mathscr{R}$ *and* $f \in L^1(G)$ *is* S'*-dependent, where* $S \cap S' = \emptyset$, *then* $\Lambda(w_s \otimes f) = w_S \otimes \bar{f}$ *for some* S'*-dependent function* $\bar{f}$ *in* $L^1(G)$ *satisfying* $\|\bar{f}\|_1 \leqq K_\tau[\tau; S]\, \|f\|_1$.

For any well-founded tree T on $\mathbf{N}$, define the operator Λ_T on $L^1(G)$ by $\Lambda_T = \mathscr{E}_T \circ \Lambda = \Lambda \circ \mathscr{E}_T$.

It is clear from Proposition 12 (2) that Λ_T is the identity on X^1_T. In order to prove Proposition 5 and hence conclude the proof of the result stated in the abstract, it remains to establish the following fact:

PROPOSITION 13. *If* $\Gamma : L^1(\lambda) \to L^1(G)$ *is a non-representable operator, then* $(I - \Lambda_T) \circ \Gamma$ *is also non-representable, for any well-founded tree* T.

Since $G = \{1, -1\}^{\mathcal{R}}$, application of Corollary 11 yields a sequence (S_k) of disjoint finite subsets of $\mathcal{R}$, so that for all k the operator $\Phi_k \circ \Gamma$ is non-representable, where $\Phi_k = \prod_{l=1}^{k} (I - \mathscr{E}_{\mathcal{R} \setminus S_l})$.

Assume $(I - \Lambda_T) \circ \Gamma$ representable and take $\varepsilon = 1/2K_\tau$. Then

LEMMA 14. *For each k, there is some $S \in \mathscr{S}_\varepsilon$ such that $S \subset T$ and $S \cap S_l \neq \emptyset$ for each $l = 1, \cdots, k$.*

Once this is obtained, we may apply Lemma 6, taking the sequence $(S_k \cap T)$ into account. This leads to a sequence (n_i) of integers with $(n_1, n_2, \cdots, n_j) \in T$ for each j, contradicting the assumption that T was well-founded. So it remains to prove the above lemma.

PROOF OF LEMMA 14. Fix k, consider the set

$$\mathscr{F} = \{S \subset \bar{T};\, S \subset \bigcup_{l=1}^{k} S_l \text{ and } S \cap S_l \neq \emptyset \text{ for each } l = 1, \cdots, k\}$$

and denote its cardinality by r.

Since $\Phi_k \circ \Gamma$ is non-representable and, by hypothesis, $\Phi_k \circ (I - \Lambda_T) \circ \Gamma = (I - \Lambda_T) \circ \Phi_k \circ \Gamma$ is representable, the operators $\Lambda_T\, \Phi_k\, \Gamma$ and hence $\mathscr{E}_T\, \Phi_k\, \Gamma$ are non-representable. Therefore, there exists some $\varphi \in L^1(\lambda)$ satisfying

$$\| \mathscr{E}_T \Phi_k \Gamma(\varphi) \| = 1 \quad \text{and} \quad \| (I - \Lambda_T) \Phi_k \Gamma(\varphi) \| < 1/3r.$$

Define $f = \mathscr{E}_T \Phi_k \Gamma(\varphi)$, which is clearly of the following form:

$$f = \sum_{S \in \mathscr{F}} w_S \otimes f_S$$

where each f_S is $(T \setminus \bigcup_{l=1}^{k} S_l)$-dependent.

Moreover, by construction, $\|f\| = 1$ and $\|(I - \Lambda) f\| < 1/3r$. By Proposition 12 (3), we see that $\Lambda(w_S \otimes f_S) = w_S \otimes \bar{f}_S$ for some $(T \setminus \bigcup_{l=1}^{k} S_l)$-dependent function $\bar{f}_S$ in $L^1(G)$ satisfying $\|\bar{f}_S\|_1 \leqq K_\tau[\tau; S] \|f_S\|_1$. Thus

$$\Lambda(f) = \sum_{S \in \mathscr{F}} w_S \otimes \bar{f}_S \quad \text{and} \quad f - \Lambda(f) = \sum_{S \in \mathscr{F}} w_S \otimes (f_S - \bar{f}_S).$$

For each $S \in \mathscr{F}$, we have

$$1/3r > \|f - \Lambda(f)\|_1 \geqq \|f_S - \bar{f}_S\|_1 \geqq \|f_S\|_1 - \|\bar{f}_S\|_1 \geqq (1 - K_\tau[\tau; S]) \|f_S\|_1$$

and hence for $S \in \mathscr{F} \setminus \mathscr{S}_\varepsilon$, by the choice of ε, $\|f_S\|_1 < 2/3r$.

Suppose $\mathscr{F} \cap \mathscr{S}_\varepsilon = \emptyset$. Then it would follow that

$$1 = \|f\|_1 \leqq \sum_{S \in \mathscr{F}} \|f_S\|_1 < 2|\mathscr{F}|/3r,$$

a contradiction. Consequently, $\mathscr{F} \cap \mathscr{S}_\varepsilon \neq \emptyset$, completing the proof.

7. Remarks and problems

(1) One can show that the operators Λ_T for T well-founded do not fix an L^1-copy and hence also satisfy condition (a) of Theorem 1. The proof of this fact is essentially contained in [4].

(2) As far as we know the $\mathscr{L}^1$-spaces constructed here are also the first examples of non-Schur $\mathscr{L}^1$-spaces which do not contain a copy of L^1.

(3) Related to this work and also [12] is the following question:

Does the class of separable Schur $\mathscr{L}^1$-spaces admit an universal element? And its weaker version:

Does there exist a separable Banach space not containing L^1 which is universal for the latter class of spaces?

References

1. D. Alspach, P. Enflo and E. Odell, *On the structure of separable $\mathscr{L}^p$-spaces* $(1 < p < \infty)$, Studia Math. **60** (1977), 79–80.
2. J. Bourgain, *A counterexample to the l^1-complementation problem*, to appear in Compositio Math.
3. J. Bourgain, *On separable Banach spaces universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. **79** (1980), 241–246.
4. J. Bourgain, *New classes of $\mathscr{L}^p$-spaces* $(1 \leqq p \leqq \infty)$, to appear in Springer Lecture Notes.
5. J. Bourgain and F. Delbaen, *A class of special $\mathscr{L}^\infty$-spaces*, Acta Math., to appear.
6. J. Bourgain, H. P. Rosenthal and G. Schechtman, *An ordinal L^p-index for Banach spaces, with application to complemented subspaces of L^p*, to appear.
7. C. Dellacherie, *Les dérivations en théorie descriptive des enesembles et le théorème de la borne*, Séminaire de Prob. XI, Univ. de Strassbourg, Lecture Notes, Vol. 581, Springer, 1977, pp. 34–46.
8. J. Diestel and J. J. Uhl, *The theory of vector measures*, Math. Surveys, No. 15, 1977.
9. P. Enflo and T. Starbird, *Subspaces of L^1 containing L^1*, to appear in Studia Math. **65**.
10. J. L. B. Gamlen and R. J. Gaudet, *On subsequences of the Haar system in $L^p[0,1]$*, $(1 < p < \infty)$, Israel J. Math. **15** (1973), 404–413.
11. A. M. Garsia, *Martingale Inequalities*, Seminar notes on recent progress, Benjamin, Reading, Mass., 1973.
12. W. B. Johnson and J. Lindenstrauss, *Examples of $\mathscr{L}^1$-spaces*, to appear.
13. W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. **217** (1979).
14. N. Kalton, *Embedding L^1 in a Banach lattice*, to appear.

15. J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in $\mathscr{L}^p$-spaces and their applications*, Studia Math. **29** (1968), 275–326.

16. J. Lindenstrauss and H. P. Rosenthal, *The $\mathscr{L}^p$-spaces*, Israel J. Math. **7** (1969), 325–349.

17. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Springer-Verlag, Berlin, 1977.

18. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Springer-Verlag, Berlin, 1979.

19. H. P. Rosenthal, *On the subspaces of L^p $(p>2)$ spanned by sequences of independent random variables*, Israel J. Math. **8** (1970), 273–303.

20. H. P. Rosenthal, *On the span in L^p of sequences of independent random variables* (*II*), Proc. 6th Berkeley Symp. on Prob. and Stat., Berkeley, CA, 1971.

21. H. P. Rosenthal, *Convolution by a Biased Coin*, The Altgeld Book 1975/76, University of Illinois, Functional Analysis Seminar.

22. G. Schechtman, *Examples of $\mathscr{L}^p$-spaces*, Israel J. Math. **22** (1975), 138–147.

23. G. Schechtman, *On Pelczynski's paper "Universal bases"*, Israel J. Math. **22** (1975), 181–184.

Vrije Universiteit Brussel
Pleinlaan 2
F7 1050 Brussels, Belgium